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J. Math. Anal. Appl. 337 (2008) 1048–1052

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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Boundedness of rough singular integral operators on the Triebel–Lizorkin spaces

Jiecheng Chen [1], Chunjie Zhang [*,2]

Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, China 310027

Received 29 January 2007

Available online 24 April 2007

Submitted by L. Grafakos

Abstract

We consider the singular integral operator T with kernel $K(x) = \Omega(x)/|x|^n$ and prove its boundedness on the Triebel–Lizorkin spaces $\dot{F}_p^{\beta,q}$ provided that Ω satisfies a size condition which contains the case $\Omega \in L^r(S^{n-1})$, $r > 1$.

Keywords: Singular integral; Triebel–Lizorkin space; Rough kernel

1. Introduction

The singular integral operator with rough kernel is defined by

$$T_{b,\alpha} f(x) = P.V. \int_{R^n} \frac{b(|y|)\Omega(y')}{|y|^{n+\alpha}} f(x-y)\, dy$$

where $\alpha \geqslant 0$, $b(s) \in L^\infty(R^+)$ and $\Omega(y') \in L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(y')\, dy' = 0.$$

We write T_b when $\alpha = 0$ and T if further $b \equiv 1$. The boundedness of the singular integral operators on various function spaces has been widely investigated and a large number of results are founded. Recently, there is an increasing interest in the study of $T_{b,\alpha}$ on the Triebel–Lizorkin spaces, see [1–4].

* Corresponding author.
E-mail address: purezhang@zju.edu.cn (C. Zhang).

1 Supported by NSFC (No. 10571156), RSDP (No. 20030335019) and ZJNSF (RC97017).
2 Supported partially by NSFC (No. 10601046).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.04.026

We first recall the definition of the homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{\beta,q}$. Let $\Phi \in C_c^\infty(R^n)$ which satisfies $\operatorname{supp}(f) \subset \{\xi\colon 1/2 < |\xi| < 2\}$ and $\Phi(\xi) > 1$ if $3/5 < |\xi| < 5/3$. Define $\Psi_k(x)$ by $\hat{\Psi}_k(\xi) = \Phi(2^k\xi)$. Then we say that a tempered distribution f belongs to $\dot{F}_p^{\beta,q}$, $\beta \geqslant 0$, $1 < p, q < \infty$ if and only if

$$\|f\|_{\dot{F}_p^{\beta,q}} = \left\| \left(\sum_k |2^{-k\beta}\Psi_k * f|^q \right)^{1/q} \right\|_{L^p} < +\infty.$$

From [6] we know that $\dot{F}_p^{0,2} = H^p$ when $0 < p \leqslant 1$, $\dot{F}_p^{0,2} = L^p$ when $1 < p < \infty$ and $\dot{F}_p^{\beta,2} = \dot{L}_\beta^p$ are the homogeneous Sobolev spaces.

In [3], the authors proved that $T_{b,\alpha}$ is bounded from $\dot{F}_p^{\alpha+\beta,q}$ to $\dot{F}_p^{\beta,q}$ when Ω belongs to $H^r(S^{n-1})$, $r = \frac{n-1}{n-1+\alpha}$, satisfying some cancellation property. Several similar results related to the maximal singular integral operators $T_{b,\alpha}^*$ were also obtained in those literatures. When $\alpha = 0$, it was proved that $\Omega \in L^r(S^{n-1})$, $r > 1$ and $\int \Omega = 0$ are sufficient to guarantee the boundedness of T_b on $\dot{F}_p^{\beta,q}$. It is natural to ask whether we can use a weaker condition $\Omega \in H^1(S^{n-1})$. Unfortunately, yet we are not able to prove this though we believe it is right. However, in this paper we shall place another assumption on Ω, that is

$$\sup_{\theta \in S^{n-1}} \int\limits_{S^{n-1}} \left|\Omega(y')\right| \left(\ln |y' \cdot \theta|^{-1}\right)^{1+\alpha} dy' < \infty, \quad \forall \alpha > 0. \tag{1}$$

This condition was introduced by Grafakos and Stefanov in [7]. It is known that the condition contains the case $\Omega \in L^r(S^{n-1})$, $r > 1$ but does differ from $H^1(S^{n-1})$, see the example constructed by Grafakos and Stefanov in [7] where the authors proved the boundedness of T on $L^p(R^n)$ with the above assumption (1).

In this paper we shall always assume, for simplicity $b(s) \equiv 1$ and use a different approach from that of [2] or [3] to prove the boundedness of T on $\dot{F}_p^{\beta,q}$. Our main result is

Theorem 1. *Let Ω belong to $L^1(S^{n-1})$ with means value zero and satisfy the assumption* (1) *for all $\alpha > 1$. Then T is bounded on $\dot{F}_p^{\beta,q}$ for $\beta \in R$ and $1 < p, q < \infty$.*

Our approach is to use an idea by Hofmann in [8] to establish a weighted norm inequality of T. As a consequence, we get a vector valued estimate from which our theorem follows immediately.

2. Some lemmas

Define

$$\sigma_k(x) = \frac{\Omega(x')}{|x|^n} \chi_{\{2^k < |x| \leqslant 2^{k+1}\}}(x), \qquad k \in Z,$$

for $\Omega \in L^1(S^{n-1})$. It is easy to see that $\{\|\sigma_k\|_{L^1}\}$ is uniformly bounded. Furthermore, we have the following estimate:

$$\left|\hat{\sigma}_k(\xi)\right| \leqslant C \min\left\{\left(\ln |2^k\xi|\right)^{-(1+\alpha)}, |2^k\xi|\right\}, \quad \forall \alpha > 1. \tag{2}$$

This conclusion can be calculated directly from the assumption (1), see [7] for details. Using $\{\sigma_k\}$, we define a maximal function associated to Ω by

$$M^\Omega f(x) = \sup_{k \in Z} \left(|\sigma_k| * |f|(x)\right).$$

We ought to write $\tilde{\sigma}_k$ and $M^{\tilde{\Omega}}$ when $\Omega(x)$ is replaced by $\tilde{\Omega}(x) = \Omega(-x)$. But since they cause no confusion, we shall abuse the notations and always write σ_k and M^Ω instead.

Lemma 1. *Suppose that $\Omega \in L^1(S^{n-1})$ has mean value zero and satisfies the assumption* (1) *for some $\alpha > 1$. Then $M^\Omega f(x)$ is bounded on $L^p(R^n)$ for $(2+\alpha)/(1+\alpha) < p < 2+\alpha$.*

This lemma can be deduced from the proof of Theorem 1 in [7].

Assuming $\Omega \in L^r(S^{n-1})$, Hofmann in [8] obtained the following weighted norm inequality,

$$\int_{R^n} \left|Tf(x)\right|^p \omega(x)\,dx \leqslant C \int_{R^n} \left|f(x)\right|^p M_s M_s^{\Omega} \omega(x)\,dx, \quad 1 < p < \infty, \tag{3}$$

for any $s > 1$. Here M is the Hardy–Littlewood maximal function, $M_s\omega = (M(\omega^s))^{1/s}$ and $\omega \in A_1$ is a Muchkenhoupt weight. Our next lemma is an analogue of (3). But we use the weaker condition (1) to replace the condition $\Omega \in L^r(S^{n-1})$.

Lemma 2. *Suppose that $\Omega \in L^1(S^{n-1})$ has mean value zero and satisfies the assumption* (1) *for all $\alpha > 0$. Let $\omega(x) \in A_1$. We then have*

$$\int_{R^n} \left|Tf(x)\right|^p \omega(x)\,dx \leqslant C_{p,s} \int_{R^n} \left|f(x)\right|^p M_s M_s^{\Omega} \omega(x)\,dx, \quad 1 < p < \infty, \tag{4}$$

for any $s > 1$.

Proof. We shall follow the method in [5] and [8]. Take $\phi \in C_c^{\infty}(R^n)$ such that $0 < \phi \leqslant 1$ and $\operatorname{supp}(\phi) \subset \{x\colon 1/2 \leqslant |x| \leqslant 2\}$. For $f \in \mathcal{S}(R^n)$, define $S_j f = \psi_j * f$ where $\hat{\psi}_j(\xi) = \phi(2^j\xi)$. In fact we can further let

$$\sum_{j=-\infty}^{\infty} \phi^3(2^j\xi) = 1.$$

Thus

$$Tf = \sum_j \sum_k S_{j+k}^3(\sigma_k * f) = \sum_j T_j f.$$

By Minkowski's inequality, it suffices to show

$$\left\|T_j f(x)\right\|_{L^p(\omega)} \leqslant C\big(1+|j|\big)^{-\delta(p,s)} \|f\|_{L^p(M_s M_s^{\Omega}\omega)} \tag{5}$$

with δ being strictly larger than 1. To do so, we need some weighted Littlewood–Paley estimates obtained by Hofmann in [8]. They are

$$\left\|\left(\sum_j |S_j h|^2\right)^{1/2}\right\|_{L^p(v)} \leqslant C\|h\|_{L^p(v)} \tag{6}$$

and

$$\left\|\sum_j S_j^2 h_j\right\|_{L^p(v)} \leqslant C\left\|\left(\sum_j |S_j h_j|^2\right)^{1/2}\right\|_{L^p(v)} \tag{7}$$

for all $1 < p < \infty$ and $v \in A_p$.

Now let us check inequality (5). By Plancherel's Theorem and estimate (2), it is not hard to obtain

$$\int_{R^n} |\sigma_k * S_{j+k} h|^2\,dx \leqslant C\big(1+|j|\big)^{-2(1+\alpha)} \int_{R^n} |h|^2\,dx, \quad \forall \alpha > 0. \tag{8}$$

On the other hand, for any $s > 1$, since $\|\sigma_k\|_{L^1} \leqslant C$, we have

$$\begin{aligned} \int_{R^n} |\sigma_k * S_{j+k} h|^2 \omega^s\,dx &\leqslant C \int_{R^n} |\sigma_k| * |\psi_{j+k}| * |h|^2 \omega^s\,dx \\ &\leqslant \int_{R^n} |h|^2 |\psi_{j+k}| * |\sigma_k| * \omega^s\,dx \\ &\leqslant \int_{R^n} |h|^2 M M^{\Omega} \omega^s\,dx. \end{aligned} \tag{9}$$

Interpolation with change of measure between (8) and (9) yields

$$\int_{R^n} \left|\sigma_k * S_{j+k} h\right|^2 \omega\, dx \leqslant C\big(1+|j|\big)^{-2(1+\alpha)(1-1/s)} \int_{R^n} |h|^2 M_s M_s^{\Omega} \omega\, dx.$$

Taking $h = S_{j+k} f$, we get

$$\int \left|\sigma_k * S_{j+k}^2 f\right|^2 \omega\, dx \leqslant C\big(1+|j|\big)^{-2(1+\alpha)(1-1/s)} \int |S_{j+k} f|^2 M_s M_s^{\Omega} \omega\, dx. \tag{10}$$

Applying the weighted Littlewood–Paley estimates (6) and (7), we have

$$\begin{aligned}
\|T_j f\|_{L^2(\omega)}^2 &= \int_{R^n} \left|\sum_k S_{j+k}^3 \sigma_k * f\right|^2 \omega\, dx \leqslant C \int_{R^n} \sum_k \left|S_{j+k}^2 \sigma_k * f\right|^2 \omega\, dx \\
&\leqslant C\big(1+|j|\big)^{-2(1+\alpha)(1-1/s)} \int_{R^n} \sum_k |S_{j+k} f|^2 M_s M_s^{\Omega} \omega\, dx \\
&\leqslant C\big(1+|j|\big)^{-2(1+\alpha)(1-1/s)} \|f\|_{L^2(M_s M_s^{\Omega} \omega)}^2
\end{aligned} \tag{11}$$

where we have also used the fact that $M_s M_s^{\Omega} \omega \in A_1 \subset A_p$ in the last inequality. If we can prove

$$\int_{R^n} \left|T_j f(x)\right|^q \omega(x)\, dx \leqslant C \int_{R^n} \left|f(x)\right|^q M_s M_s^{\Omega} \omega(x)\, dx, \quad 1 < q < \infty, \tag{12}$$

then another interpolation between (11) and (12) yields inequality (5). In fact for any fixed $1 < p < \infty$, since assumption (1) holds for all $\alpha > 0$, we may take α sufficiently large such that $\delta(p, s) > 1$. To get (12), we need the following Lemma 3.

Lemma 3. *Let Ω and ω be as in Lemma* 2. *Then*

$$\left\| \left(\sum_k |\sigma_k * g_k|^2 \right)^{1/2} \right\|_{L^q(\omega)} \leqslant C \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_{L^q(M_s^{\Omega} \omega)}, \quad 1 < q < \infty, \tag{13}$$

holds for any $s > 1$.

We shall continue to prove Lemma 2 and put the proof of Lemma 3 afterward. By inequality (7), we see that $\|T_j f\|_{L^q(\omega)}$ is bounded by

$$C \left\| \left(\sum_k \left|\sigma_k * S_{j+k}^2 f\right|^2 \right)^{1/2} \right\|_{L^q(\omega)}$$

which, by applying Lemma 3, does not exceed

$$C \left\| \left(\sum_k \left|S_{j+k}^2 f\right|^2 \right)^{1/2} \right\|_{L^q(M_s^{\Omega} \omega)}.$$

Noting that $M_s^{\Omega} \omega \leqslant M_s M_s^{\Omega} \omega \in A_1$, we then reach (12) by another application of (6). □

Now let us sketch the proof of Lemma 3 briefly. It is a weighted version of the lemma in [5] which was also proved in [8] under the condition that $\Omega \in L^r(S^{n-1})$, $r > 1$. In fact, the key point in getting (13) is the boundedness of M^{Ω} on any L^p, $p > 1$. But this can be reached by applying Lemma 1 as well as the assumption in (1) that α might be chosen sufficiently large. Repeating the argument of Lemma 2.9 in [8] we get our Lemma 3.

Remark 1. In Lemma 2, if condition (1) is merely satisfied for some fixed $\alpha > 0$, then by checking carefully the proof of Lemma 2.9 in [8], we can show that (13) is valid only when $p \in (1, 2] \cup (\frac{2(s-1)}{s(1+\alpha)} + 2, 4 + 2\alpha - \frac{2+2\alpha}{s})$, $s > 1$. This is due to a restriction in Lemma 1 that M^{Ω} is bounded on L^p for those $p \in (\frac{2+\alpha}{1+\alpha}, 2+\alpha)$ rather than the whole range. Respectively, the range of p such that (4) holds can be calculated for each $s > 1 + 1/\alpha$ (this restriction is caused by (11) where $(1+\alpha)(1-1/s)$ must be strictly larger than 1).

3. Proof of the theorem

Proof of Theorem 1. With Lemma 2 in hand, it is routine to check that T satisfies the following vector valued inequality:

$$\left\| \left(\sum_j |Tf_j|^q \right)^{1/q} \right\|_{L^p} \leqslant C \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L^p}, \quad 1 < q, p < \infty.$$

Now by the definition of $\dot{F}_p^{\beta,q}$,

$$\begin{aligned} \|Tf\|_{\dot{F}_p^{\beta,q}} &= \left\| \left(\sum_k \left| 2^{-k\beta} \Psi_k * Tf \right|^q \right)^{1/q} \right\|_{L^p} \\ &= \left\| \left(\sum_k \left| T\left(2^{-k\beta} \Psi_k * f\right) \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leqslant C \left\| \left(\sum_k \left| 2^{-k\beta} \Psi_k * f \right|^q \right)^{1/q} \right\|_{L^p} \\ &= C \|f\|_{\dot{F}_p^{\beta,q}}. \qquad \square \end{aligned}$$

Remark 2. Our proof might be carried out on any convolutional operator once we get a weighted norm inequality for it.

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